The Monotonicity of the Gravitational Entropy Scalar within Quiescent Cosmology

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Abstract. In this paper we show that Quiescent Cosmology [1, 2, 3] is consistent with Penrose's Weyl Curvature Hypothesis and the notion of gravitational entropy [4]. Gravitational entropy, from a conceptual point of view, acts in an opposite fashion to the more familiar notion of entropy. A closed system of gravitating particles will coalesce whereas a collection of gas particles will tend to diffuse; regarding increasing entropy, these two scenarios are identical. What has been shown previously [2, 3] is that gravitational entropy at the initial singularity predicted by Quiescent Cosmology - the Isotropic Past Singularity (IPS) - tends to zero. The results from this paper show that not only is this the case but that gravitational entropy increases as this singularity evolves.

In the first section of this paper we present relevant background information and motivation. In the second section of this paper we present the main results of this paper. Our third section contains a discussion of how this result will inspire future research before we make concluding remarks in our final section.

1. Background and Motivation

1.1. Quiescent Cosmology

Barrow introduced the world to Quiescent Cosmology in 1978 [1] as an attempt to explain the current large scale isotropy and homogeneity of the Universe. Quiescent Cosmology effectively states that the Universe began in a highly ordered state and has evolved away from its highly regular and smooth beginning because of gravitational attraction ‡. This means that the reason we continue to observe large scale regularity is because we exist in an early stage of cosmological evolution. In order for Quiescent Cosmology to be compatible with a Big Bang type singularity, it is necessary that that singularity is one that is isotropic. This type of initial isotropic singularity was given a rigorous mathematical definition in 1985 by Goode and Wainwright [2] using a conformal relationship between two spacetimes. The definition given in this paper is due to Scott [6] who removed the inherent technical redundancies of the original definition.

Goode and Wainwright based their analysis on the beginning of the universe, but recently there has been increasing interest in possible future evolutions of the Universe. Höhn and Scott [3] introduced different isotropic and anisotropic definitions that describe possible future end states of the Universe. Following Goode and Wainwright they also exploited conformal relationships between spacetimes.

1.2. Conformal Structures

In this paper we primarily deal with isotropic structures and thus we require the conformal definitions that relate to isotropic initial and final states of the universe. In order for this paper to be fully appreciated, however, results pertaining to isotropic structures will be put in context with anisotropic structures; these definitions will also be presented in this introduction.

The isotropic definitions comprise of the Isotropic Past Singularity, the Isotropic

‡ This is in contrast to the ideals of Chaotic Cosmology, made famous by Misner [5]

Future Singularity and the Future Isotropic Universe. The anisotropic definitions of Quiescent Cosmology are the Anisotropic Future Endless Universe and the Anisotropic Future Singularity. Any ancillary definitions that are needed will also be included.

Definition 1 (Conformally related metric) A metric $\mathbf{g}$ is said to be conformally related to a metric $\tilde{\mathbf{g}}$ on a manifold $\mathcal{M}$ if there exists a conformal factor Ω such that

$$\mathbf{g} = \Omega^2 \tilde{\mathbf{g}}, \text{ where } \Omega \text{ is a strictly positive function on } \mathcal{M}. \tag{1}$$

Definition 2 (Cosmic time function) For a space-time $(\mathcal{M}, g)$, a cosmic time function is a function T on the manifold $\mathcal{M}$ which increases along every future-directed causal curve.

1.2.1. Isotropic Definitions It should be noted that we will henceforth denote relevant quantities for past cosmological frameworks with a tilde ($\sim$) and for future cosmological frameworks we will use a bar ($-$).

Definition 3 (Isotropic Past Singularity (IPS)) A space-time $(\mathcal{M}, \mathbf{g})$ admits an Isotropic Past Singularity if there exists a space-time $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$, a smooth cosmic time function T defined on $\tilde{\mathcal{M}}$ and a conformal factor $\Omega(T)$ which satisfy

i) $\mathcal{M}$ is the open submanifold $T > 0$,

ii) $\mathbf{g} = \Omega^2(T)\tilde{\mathbf{g}}$ on $\mathcal{M}$, with $\tilde{\mathbf{g}}$ regular (at least C^3 and non-degenerate) on an open neighbourhood of $T = 0$,

iii) $\Omega(0) = 0$ and $\exists\, b > 0$ such that $\Omega \in C^0[0, b] \cap C^3(0, b]$ and $\Omega(0, b] > 0$,

iv) $\lambda \equiv \lim_{T\to 0^+} L(T)$ exists, $\lambda \neq 1$, where $L \equiv \frac{\Omega''}{\Omega}\left(\frac{\Omega}{\Omega'}\right)^2$ and a prime denotes differentiation with respect to T.

It was demonstrated by Goode and Wainwright [2] that, in order to ensure initial asymptotic isotropy, it is also necessary to introduce a constraint on the cosmological fluid flow.

Definition 4 (IPS fluid congruence) With any unit timelike congruence $\mathbf{u}$ in $\mathcal{M}$ we

can associate a unit timelike congruence $\tilde{\mathbf{u}}$ in $\tilde{\mathcal{M}}$ such that

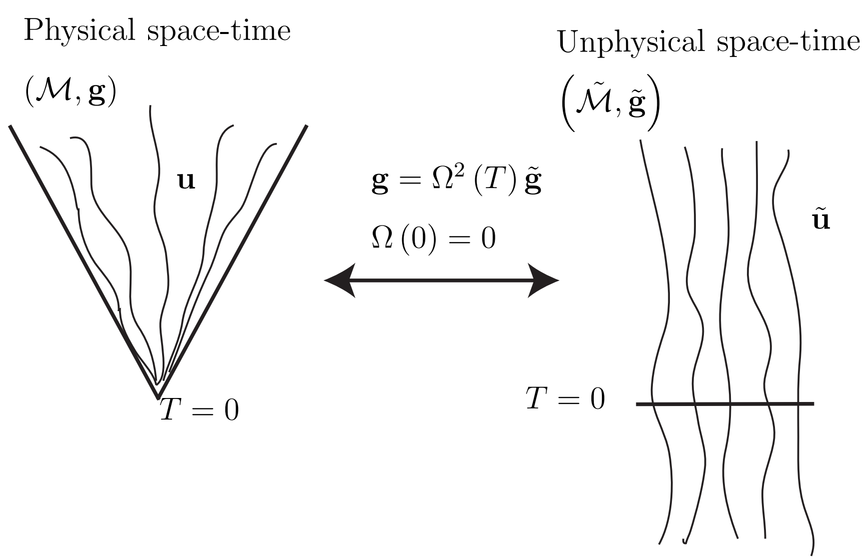
$$\tilde{\mathbf{u}} = \Omega \mathbf{u} \qquad \text{in } \mathcal{M}. \tag{2}$$

a) If we can choose $\tilde{\mathbf{u}}$ to be regular (at least C^3) on an open neighbourhood of $T = 0$ in $\tilde{\mathcal{M}}$, we say that $\mathbf{u}$ is regular at the IPS.

b) If, in addition, $\tilde{\mathbf{u}}$ is orthogonal to $T = 0$, we say that $\mathbf{u}$ is orthogonal to the IPS.

In figure 1 we present a pictorial interpretation of the IPS.

Figure 1. A pictorial interpretation of an IPS. The fluid flow is represented by $\mathbf{u}$.



Below is given the analogous definition of an Isotropic Future Singularity introduced by Höhn and Scott [3], followed by the constraint on the fluid flow required to ensure final asymptotic isotropy. The IFS is not compatible with the fundamental ideals of Quiescent Cosmology (that the end state of the Universe is anisotropic) but it remains a structure worth analysing.

Definition 5 (Isotropic Future Singularity (IFS)) A space-time $(\mathcal{M}, \mathbf{g})$ admits an Isotropic Future Singularity if there exists a space-time $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$, a smooth cosmic time function $\bar{T}$ defined on $\bar{\mathcal{M}}$, and a conformal factor $\bar{\Omega}(\bar{T})$ which satisfy

i) $\mathcal{M}$ is the open submanifold $\bar{T} < 0$,

ii) $\mathbf{g} = \bar{\Omega}^2(\bar{T})\bar{\mathbf{g}}$ on $\mathcal{M}$, with $\bar{\mathbf{g}}$ regular (at least C^2 and non-degenerate) on an open neighbourhood of $\bar{T} = 0$,

iii) $\bar{\Omega}(0) = 0$ and $\exists\ c > 0$ such that $\bar{\Omega} \in C^0[-c, 0] \cap C^2[-c, 0)$ and $\bar{\Omega}$ is positive on $[-c, 0)$,

iv) $\bar{\lambda} \equiv \lim_{\bar{T}\to 0^-} \bar{L}(\bar{T})$ exists, $\bar{\lambda} \neq 1$, where $\bar{L} \equiv \frac{\bar{\Omega}''}{\bar{\Omega}}\left(\frac{\bar{\Omega}}{\bar{\Omega}'}\right)^2$ and a prime denotes differentiation with respect to $\bar{T}$.

Definition 6 (IFS fluid congruence) With any unit timelike congruence $\mathbf{u}$ in $\mathcal{M}$ we can associate a unit timelike congruence $\bar{\mathbf{u}}$ in $\bar{\mathcal{M}}$ such that

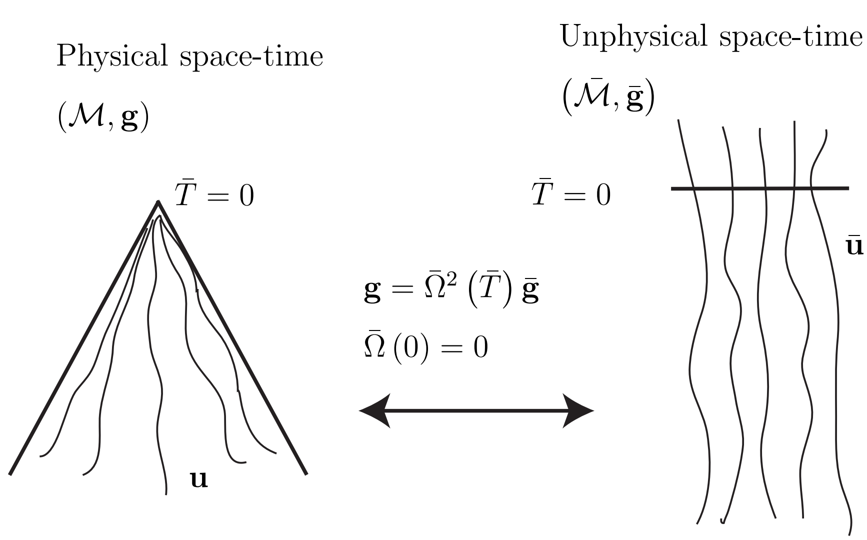
$$\bar{\mathbf{u}} = \bar{\Omega}\mathbf{u} \qquad \text{in } \mathcal{M}. \tag{3}$$

a) If we can choose $\bar{\mathbf{u}}$ to be regular (at least C^2) on an open neighbourhood of $\bar{T} = 0$ in $\bar{\mathcal{M}}$, we say that $\mathbf{u}$ is regular at the IFS.

b) If, in addition, $\bar{\mathbf{u}}$ is orthogonal to $\bar{T} = 0$, we say that $\mathbf{u}$ is orthogonal to the IFS.

In figure 2 we present a pictorial interpretation of the IFS.

Figure 2. A pictorial interpretation of an IFS. The fluid flow is represented by $\mathbf{u}$. It can be seen that an IFS is essentially a time reversal of an IPS.



Finally we give below the definition of a Future Isotropic Universe introduced by Höhn and Scott [3]. This definition covers the further possibility for a conformal structure with an isotropic future behaviour, which does not necessarily lead to a future singularity; for example, some open FRW universes satisfy this definition [7].

Definition 7 (Future Isotropic Universe (FIU)) A space-time $(\mathcal{M}, \mathbf{g})$ is said to be a Future Isotropic Universe if there exists a space-time $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$, a smooth cosmic time function $\bar{T}$ defined on $\bar{\mathcal{M}}$, and a conformal factor $\bar{\Omega}(\bar{T})$ which satisfy

i) $\lim\limits_{\bar{T}\to 0^-} \bar{\Omega}(\bar{T}) = +\infty$ and $\exists\ c > 0$ such that $\bar{\Omega} \in C^2[-c, 0)$ and $\bar{\Omega}$ is strictly monotonically increasing and positive on $[-c, 0)$,

ii) $\bar{\lambda}$ as defined above exists, $\bar{\lambda} \neq 1, 2$, and $\bar{L}$ is continuous on $[-c, 0)$ and

iii) otherwise the conditions of definitions 5 and 6 are satisfied.

1.2.2. Anisotropic Definitions All anisotropic definitions§ refer to the future and hence their unphysical quantities are denoted by a bar $(-)$. It is conceivable, however, that these definitions could be recast for past cosmological states (similar to the IPS/IFS scenarios).

Definition 8 (Causal Degeneracy) Consider $p \in \mathcal{M}$. Let $\gamma_p(s)$ be a future inextendible causal curve such that $\gamma_p(s) : [0, a) \to \mathcal{M}$, where $a \in \mathbb{R}^+ \cup \{\infty\}$, such that $p = \gamma_p(a) \equiv \lim\limits_{s\to a} \gamma_p(s)$ with limiting tangent vector $\gamma_p' \neq 0$ at p.The metric $\bar{g}$ is said to be causally degenerate at p if there exists a curve γ_p which satisfies $\bar{g}(\gamma_p', X) = 0\ \forall\ X \in T_p\bar{M}$. (Note that this assumes the metric is continuous on an open neighbourhood of p).

Definition 9 (Anisotropic Future Endless Universe (AFEU)) A spacetime, $(\mathcal{M}, g)$ is said to be an Anisotropic Future Endless Universe if there exists

(i) a larger manifold $\bar{\mathcal{M}} \supset \mathcal{M}$,

§ The definitions given here are slightly modified from the original ones [3]. The modifications are due to the removal of the limiting causal future at $\bar{T} = 0$. This has been replaced with our open neighbourhood on $\bar{T} = 0$.

(ii) a smooth function $\bar{T}$ defined on $\bar{\mathcal{M}}$ (with $\bar{\nabla}\bar{T} \neq 0$ everywhere on $\bar{\mathcal{M}}$) such that $\mathcal{M}$ is the open submanifold $\bar{T} < 0$,

(iii) a C^0 tensor field $\bar{\mathbf{g}}$ of type $(0,2)$ defined on $\mathcal{M} \cup \mathcal{N}$, where $\mathcal{N}$ is an open neighbourhood of $\bar{T} = 0$ in $\bar{\mathcal{M}}$, and

(iv) a conformal factor $\bar{\Omega}(\bar{T})$ defined on $\mathcal{M}$, which satisfies

(a) $\bar{T}$ is a cosmic time function on $\mathcal{M} \cup \mathcal{N}$,

(b) $\mathbf{g} = \bar{\Omega}^2\left(\bar{T}\right)\bar{\mathbf{g}}$ on $\mathcal{M}$ and $\bar{\mathbf{g}}$ is degenerate on $\bar{T} = 0$,

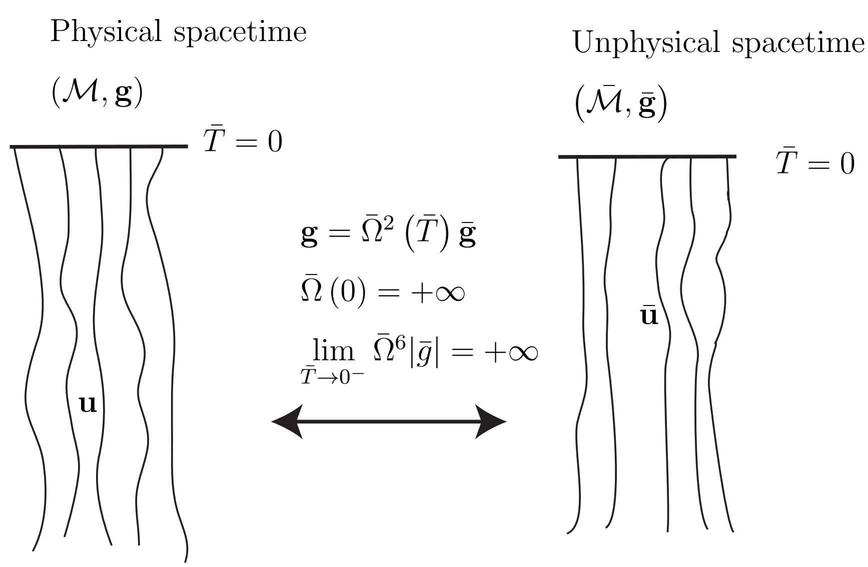
(c) $\lim_{\bar{T}\to 0^-} \bar{\Omega}\left(\bar{T}\right) = +\infty$ and $\exists\, c > 0$ such that $\bar{\Omega} \in C^2[-c, 0)$ and $\bar{\Omega}$ is strictly monotonically increasing and positive on $[-c, 0)$,

(d) $\bar{L}$ as defined above is continuous on $[-c, 0)$, $\bar{\lambda}$ exists, $\bar{\lambda} \neq 1$, and

(e) $\lim_{\bar{T}\to 0^-} \bar{\Omega}^6|\bar{g}| = \infty$, where $\bar{g}$ is the determinant of $\bar{\mathbf{g}}$.

Thus, the next figure we present is that which represents the AFEU, seen in figure 3.

Figure 3. A pictorial interpretation of an AFEU.



Definition 10 (Anistropic Future Singularity (AFS)) A spacetime, $(\mathcal{M}, g)$ is said to be an Anisotropic Future Singularity if there exists

(i) a larger manifold $\bar{\mathcal{M}} \supset \mathcal{M}$,

(ii) a smooth function $\bar{T}$ defined on $\bar{\mathcal{M}}$ (with $\bar{\nabla}\bar{T} \neq 0$ everywhere on $\bar{\mathcal{M}}$) such that $\mathcal{M}$ is the open submanifold $\bar{T} < 0$,

(iii) a C^0 tensor field $\bar{\mathbf{g}}$ of type $(0, 2)$ defined on $\mathcal{M} \cup \mathcal{N}$, where $\mathcal{N}$ is an open neighbourhood of $\bar{T} = 0$ in $\bar{\mathcal{M}}$, and

(iv) a conformal factor $\bar{\Omega}(\bar{T})$ defined on $\mathcal{M}$, which satisfies

  (a) $\bar{T}$ is a cosmic time function on $\mathcal{M} \cup \mathcal{N}$,

  (b) $\mathbf{g} = \bar{\Omega}^2\left(\bar{T}\right)\bar{\mathbf{g}}$ on $\mathcal{M}$ and $\bar{\mathbf{g}}$ is degenerate on $\bar{T} = 0$,

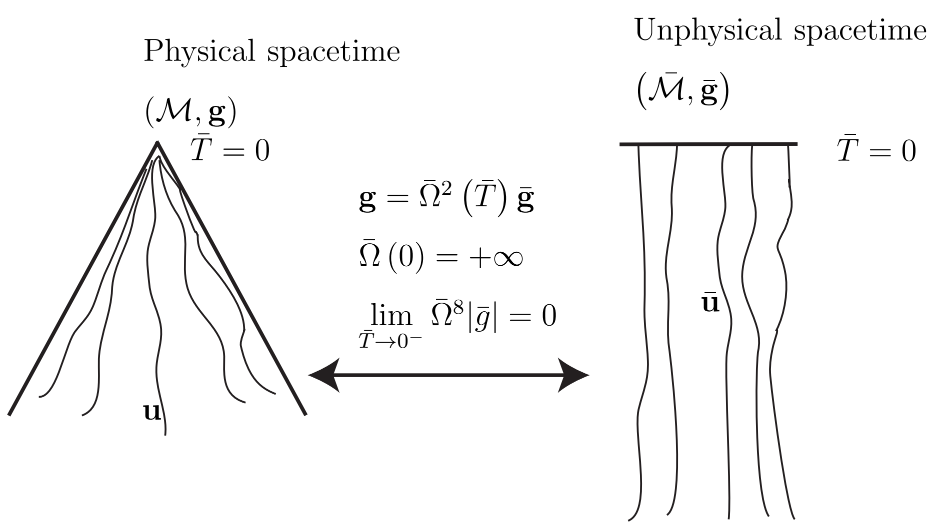
  (c) $\lim_{\bar{T}\to 0^-} \bar{\Omega}\left(\bar{T}\right) = +\infty$ and $\exists\, c > 0$ such that $\bar{\Omega} \in C^2[-c, 0)$ and $\bar{\Omega}$ is strictly monotonically increasing and positive on $[-c, 0)$,

  (d) $\bar{L}$ as defined above is continuous on $[-c, 0)$, $\bar{\lambda}$ exists, $\bar{\lambda} \neq 1$, and

  (e) $\lim_{\bar{T}\to 0^-} \bar{\Omega}^8|\bar{g}| = 0$, where $\bar{g}$ is the determinant of $\bar{\mathbf{g}}$.

Figure 4 demonstrates the degenerate nature of the AFS.

Figure 4. A pictorial interpretation of an AFS.



2. Gravitational Entropy

The Weyl Curvature Hypothesis (WCH) of Penrose [4] states that the Weyl curvature at an initial (Big Bang) singularity must be bounded and has been increasing ever since.

Gravitational entropy [8, 9, 10, 3] is closely linked to the WCH as will be explained here. A common mental image that comes to mind when thinking of entropy involves a gas expanding within a chamber - thus maximising the entropy of the system. If this imagery were applied to a collection of gravitating particles the particles would attract one another and end up in a system that seems to have less entropy than when it started. Penrose [4] addressed this problem by postulating the gravitational entropy of a system reaches a maximum when gravitational collapse results in a black hole. This means that a collection of particles that coalesce are actually increasing the entropy available and thus gravity can be consistent with thermodynamics. In General Relativity a quantity that is hypothesised to be a measure of gravitational entropy [4] is the ratio between the Weyl and Ricci curvature invariants

$$K = \frac{C^{abcd}C_{abcd}}{R^{ef}R_{ef}}. \tag{4}$$

Where it is understood that Weyl curvature describes the curvature purely due to the gravitational field and the Ricci scalar will describe the curvature due to matter. In order for entropy to have been globally increasing (as is expected) from the Big Bang, the entropy at the Big Bang must have been low, i.e. $K = 0$ at the Big Bang. The original interpretation for this was that the Weyl scalar must have been identically zero at the Big Bang but this constraint was too strict [11] as it would have ruled out all cosmological models apart from the Friedmann-Robertson-Walker (FRW) solutions. The compromise is that the Weyl scalar must be asymptotically dominated by the Ricci scalar at the Big Bang. The IPS has been shown to be consistent with this hypothesis when Goode and Wainwright [2] proved that, as an IPS is approached, $\lim_{\bar{T}\to 0} K = 0$.

2.1. The K Theorem

When Höhn and Scott [3] expanded the Quiescent Cosmology framework to consider possible future cosmological states, they were able to show that Goode and Wainwright's result is able to be expanded to all isotropic structures; this result was published as The K Theorem.

Theorem 11 (The K Theorem) Let $(\mathcal{M}, \mathbf{g})$ and $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$ be two spacetimes which

are related via the conformal structure $\mathbf{g} = \bar{\mathbf{\Omega}}^{\mathbf{2}}(\bar{\mathbf{T}})\bar{\mathbf{g}}$, where $\bar{T}$ is a smooth cosmic time function defined on $(\bar{\mathcal{M}}, \bar{\mathbf{g}})$ and $\bar{g}$ is non-degenerate and at least C^2 on an open neighbourhood of $\bar{T} = 0$. Let one of the following conditions be true

(i) $\bar{T} \to 0^-$, $\lim_{\bar{T}\to 0^-} \bar{\Omega}(0) = \infty$ and $\bar{\Omega}$ is positive, C^2 and strictly increasing on some interval $(0, c]$

(ii) $\bar{T} \to 0^-$, $\lim_{\bar{T}\to 0^-} \bar{\Omega}(0) = 0$ and $\bar{\Omega}$ is positive, C^2 and strictly decreasing on some interval $[-c, 0)$

(iii) $\bar{T} \to 0^+$, $\lim_{\bar{T}\to 0^+} \bar{\Omega}(0) = \infty$ and $\bar{\Omega}$ is positive, C^2 and strictly decreasing on some interval $(0, c]$

(iv) $\bar{T} \to 0^+$, $\lim_{\bar{T}\to 0^+} \bar{\Omega}(0) = 0$ and $\bar{\Omega}$ is positive, C^2 and strictly increasing on some interval $(0, c]$

and $\bar{\lambda} \neq 1$ then $\lim_{\bar{T}\to 0^\pm} \frac{C^{abcd}C_{abcd}}{R^{ef}R_{ef}} = 0$.

Theorem 11 is important as it shows that, with a few assumptions placed upon the conformal factor, all conformally regular spacetimes give asymptotic isotropic behaviour. This demonstrates that the Quiescent Cosmology is, at least asymptotically, consistent with Penrose's ideas about Weyl curvature at an initial singularity. The other half of Penrose's hypothesis (that gravitational entropy increases after a Big Bang) is yet to be answered. It is important to know this answer because if K is always asymptotically zero according to Quiescent Cosmology then it means that Quiescent Cosmology is not compatible with Penrose's ideas in full generality. The most telling indicator of how K may or may not increase is shown by considering the derivative with respect to cosmic time. If this value is positive as $\bar{T}$ increases from zero, it means that K will be monotonically increasing away from the IPS; this is in line with the prediction of Penrose [4].

3. The Derivative of the Gravitational Entropy Scalar

Recall that K is given by

$$K = \frac{C^{abcd}C_{abcd}}{R^{ef}R_{ef}} \tag{5}$$

$$= \frac{\bar{\Omega}^{-4}\bar{C}^{abcd}\bar{C}_{abcd}}{R^{ef}R_{ef}}. \tag{6}$$

We can now take the partial derivative of this scalar with respect to cosmic time to obtain the following

$$K' = \frac{(\bar{\Omega}^{-4}\bar{C}^{abcd}\bar{C}_{abcd})_{,m}R^{ef}R_{ef} - \bar{\Omega}^{-4}(R^{ef}R_{ef})_{,m}\bar{C}^{abcd}\bar{C}_{abcd}}{(R^{ef}R_{ef})^2} \tag{7}$$

It is clear that we will need to know the derivative of the physical Ricci scalar and the unphysical Weyl scalar; we present those now.

3.1. The Unphysical Weyl Scalar's Derivative

The derivative of the physical Weyl scalar is

$$\left(C^{abcd}C_{abcd}\right)' = -4\bar{\Omega}^{-5}\bar{C}^{abcd}\bar{C}_{abcd}\bar{T}_{,m} + \bar{\Omega}^{-4}\left(\bar{C}^{abcd}\bar{C}_{abcd}\right)'. \tag{8}$$

Where the unphysical Weyl tensor's derivative is

$$\begin{aligned}
(\bar{C}_{abcd})' = (&\bar{R}_{abcd,m} - \frac{1}{4}\bar{g}^{ij}{}_{,m}\left((\bar{g}_{ac}\bar{R}_{idjb} - \bar{g}_{ad}\bar{R}_{icjb}) - (\bar{g}_{bc}\bar{R}_{idja} - \bar{g}_{bd}\bar{R}_{icja})\right)\\
&- \frac{1}{4}\bar{g}^{ij}((\bar{g}_{ac,m}\bar{R}_{idjb} + \bar{g}_{ac}\bar{R}_{idjb,m} - \bar{g}_{ad,m}\bar{R}_{icjb} - \bar{g}_{ad}\bar{R}_{icjb,m})\\
&- (\bar{g}_{bc,m}\bar{R}_{idja} + \bar{g}_{bc}\bar{R}_{idja,m} - \bar{g}_{bd}\bar{R}_{icja} - \bar{g}_{bd,m}\bar{R}_{icja,m}))\\
&+ \frac{1}{6}(\bar{g}^{ij}{}_{,m}\bar{g}^{kl}\bar{R}_{ikjl} + \bar{g}^{ij}\bar{g}^{kl}{}_{,m}\bar{R}_{ikjl} + \bar{g}^{ij}\bar{g}^{kl}\bar{R}_{ikjl,m})\\
&\cdot\ (\bar{g}_{ac,m}\bar{g}_{db} + \bar{g}_{ac}\bar{g}_{db,m} - \bar{g}_{ad,m}\bar{g}_{cb} - \bar{g}_{ad}\bar{g}_{cb,m}))
\end{aligned} \tag{9}$$

As such the unphysical Weyl scalar's derivative is

$$\begin{aligned}
(\bar{C}^{abcd}\bar{C}_{abcd})_{,m} = (&\bar{g}^{an}{}_{,m}\bar{g}^{bo}\bar{g}^{cp}\bar{g}^{dq} + \bar{g}^{an}\bar{g}^{bo}{}_{,m}\bar{g}^{cp}\bar{g}^{dq} + \bar{g}^{an}\bar{g}^{bo}\bar{g}^{cp}{}_{,m}\bar{g}^{dq}\\
&+ \bar{g}^{an}\bar{g}^{bo}\bar{g}^{cp}\bar{g}^{dq}{}_{,m})(\bar{C}_{abcd,m}\bar{C}_{nopq} + \bar{C}_{abcd}\bar{C}_{nopq,m})
\end{aligned} \tag{10}$$

This is the easier derivative to calculate because of the simple relationship between the unphysical Weyl scalar and its physical counterpart.

3.2. The Physical Ricci Scalar's Derivative

A calculation of the physical Ricci scalar's derivative is more involved than the Weyl scalar's but the process is similar enough. With this in mind, first recall the physical Ricci scalar

$$R^{ef}R_{ef} = \bar{\Omega}^{-4}\Big(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2 - \bar{L} + 1\right)$$

$$- 2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^3 \left(4\left(2-\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij} - 2\left(4\bar{L}-1\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\right)$$
$$+ \left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^2 \Big(4\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{:ef}\bar{T}_{:ij} + 8\bar{g}^{ei}\bar{g}^{ei}\left(\bar{T}_{:ei}\right)^2 + 4\bar{g}^{ei}\bar{g}^{fj}\left(2-\bar{L}\right)\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}$$
$$- 2\bar{g}^{ei}\left(1+\bar{L}\right)\bar{R}\bar{T}_{,e}\bar{T}_{,i}\Big)$$
$$- 2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)\left(2\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{:ij} + \bar{g}^{ei}\bar{R}\bar{T}_{:ei}\right) + \bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{R}_{ij}\Big). \tag{11}$$

This means that the Ricci scalar's derivative is given by

$$\left(R^{ef}R_{ef}\right)_{,m} = \left(\bar{\Omega}^{-4}\right)_{,m}\bar{\Omega}^4 R^{ef}R_{ef}$$
$$+ \bar{\Omega}^{-4}\Big(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)$$
$$- 2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^3 \left(4\left(2-\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij} - 2\left(4\bar{L}-1\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\right)$$
$$+ \left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^2 \Big(4\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{:ef}\bar{T}_{:ij} + 8\bar{g}^{ei}\bar{g}^{ei}\left(\bar{T}_{:ei}\right)^2 + 4\bar{g}^{ei}\bar{g}^{fj}\left(2-\bar{L}\right)\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}$$
$$- 2\bar{g}^{ei}\left(1+\bar{L}\right)\bar{R}\bar{T}_{,e}\bar{T}_{,i}\Big)$$
$$- 2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)\left(2\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{:ij} + \bar{g}^{ei}\bar{R}\bar{T}_{:ei}\right) + \bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{R}_{ij}\Big)_{,m}. \tag{12}$$

This equation is simpler if we analyse it one term at a time. To aid the reader following along with this derivation, the following two equations may prove helpful

$$\left(\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^n\right)_{,m} = \left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{n+1}\left(\bar{L}-1\right)\bar{T}_{,m} \tag{13}$$
$$\bar{L}_{,m} = \bar{L}\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\left(1-2\bar{L}\right) + \frac{\bar{\Omega}'''}{\bar{\Omega}''}\right)\bar{T}_{,m} \tag{14}$$

For simplicity's sake, the derivative of the physical Ricci scalar is given one line at a time. Beginning with the top line,

$$\left(\bar{\Omega}^{-4}\right)_{,m}\bar{\Omega}^4 R^{ef}R_{ef} = -4\frac{\bar{\Omega}'}{\bar{\Omega}^5}\bar{T}_{,m}\bar{\Omega}^4 R^{ef}R_{ef} \tag{15}$$

Now the more interesting lines, beginning with the second

$$\bar{\Omega}^{-4}\left(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)\right)_{,m}$$
$$= 48\bar{\Omega}^{-4}\Big(\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^5\left(\bar{L}-1\right)\left(\bar{L}^2-\bar{L}+1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{T}_{,m}$$
$$+ 12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\Big(2\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)\left(\bar{g}^{ei}{}_{,m}\bar{T}_{,e}\bar{T}_{,i} + \bar{g}^{ei}\left(\bar{T}_{,em}\bar{T}_{,i} + \bar{T}_{,e}\bar{T}_{,im}\right)\right)$$
$$\cdot \quad \left(\bar{L}^2-\bar{L}+1\right) + \left(2\bar{L}-1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{L}_{,m}\Big)\Big), \tag{16}$$

now the third line

$$
\begin{aligned}
&\bar{\Omega}^{-4}\Big(2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{3}\Big(\left(8-4\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij}\\
&-\ \left(8\bar{L}+2\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\Big)\Big)_{,m}\\
&=6\bar{\Omega}^{-4}\Big(\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{4}\left(\bar{L}-1\right)\Big(\left(8-4\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij}\\
&-\ \left(8\bar{L}+2\right)\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\Big)\bar{T}_{,m}\\
&-2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{3}\Big(4\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij}\bar{L}_{,m}\\
&+\ \left(8-4\bar{L}\right)\Big(\left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj}+\bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij}\\
&+\bar{g}^{ei}\bar{g}^{fj}\Big(\bar{T}_{,em}\bar{T}_{,f}\bar{T}_{:ij}+\bar{T}_{,e}\bar{T}_{,fm}\bar{T}_{:ij}\\
&+\bar{T}_{,e}\bar{T}_{,f}\bar{T}_{:ij,m}\Big)\Big)-8\left(\bar{L}_{,m}\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\right)\\
&-\ \left(8\bar{L}+2\right)\Big(\left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj}+\bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj}\\
&-\bar{g}^{ei}\bar{g}^{fj}\left(\bar{T}_{,em}\bar{T}_{,i}\bar{T}_{:fj}+\bar{T}_{,e}\bar{T}_{,im}\bar{T}_{:fj}+\bar{T}_{,e}\bar{T}_{,i}\bar{T}_{:fj,m}\right)\Big)\Big)\Big),
\end{aligned}
\tag{17}
$$

the fourth line now

$$
\begin{aligned}
&\bar{\Omega}^{-4}\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{2}\Big(4\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{:ef}\bar{T}_{:ij}+8\left(\bar{g}^{ei}\bar{T}_{:ei}\right)^{2}+4\left(2-\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}\\
&-2\left(1+\bar{L}\right)\bar{g}^{ei}\bar{R}\bar{T}_{,e}\bar{T}_{,i}\Big)_{,m}\\
&=\bar{\Omega}^{-4}\Big(\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{3}\left(\bar{L}-1\right)\Big(4\bar{g}^{ei}\bar{g}^{fj}\bar{T}_{:ef}\bar{T}_{:ij}+8\left(\bar{g}^{ei}\bar{T}_{:ei}\right)^{2}\\
&+4\left(2-\bar{L}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}-2\left(1+\bar{L}\right)\bar{g}^{ei}\bar{R}\bar{T}_{,e}\bar{T}_{,i}\Big)\bar{T}_{,m}\\
&+4\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{2}\Big(\left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj}+\bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{T}_{:ef}\bar{T}_{:ij}+\bar{g}^{ei}\bar{g}^{fj}\left(\bar{T}_{:ef,m}\bar{T}_{:ij}+\bar{T}_{:ef}\bar{T}_{:ij,m}\right)\\
&+4\bar{g}^{ei}\bar{T}_{:ei}\left(\bar{g}^{ei}{}_{,m}\bar{T}_{:ei}+\bar{g}^{ei}\bar{T}_{:ei,m}\right)-\left(\bar{L}_{,m}\right)\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}\\
&+\ \left(2-\bar{L}\right)\Big(\left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj}+\bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{R}_{ef}\bar{T}_{,i}\bar{T}_{,j}\\
&+\bar{g}^{ei}\bar{g}^{fj}\Big(\bar{R}_{ef,m}\bar{T}_{,i}\bar{T}_{,j}+\bar{R}_{ef}\left(\bar{T}_{,im}\bar{T}_{,j}+\bar{T}_{,i}\bar{T}_{,jm}\right)\Big)\Big)\\
&-\frac{1}{2}\Big(\bar{L}_{,m}\bar{g}^{ei}\bar{R}\bar{T}_{,e}\bar{T}_{,i}+\left(1+\bar{L}\right)\Big(\bar{g}^{ei}{}_{,m}\bar{R}\bar{T}_{,e}\bar{T}_{,i}\\
&-\bar{g}^{ei}\Big(\left(\bar{R}_{,m}\bar{T}_{,e}\bar{T}_{,i}\right)+\bar{R}\left(\bar{T}_{,em}\bar{T}_{,i}+\bar{T}_{,e}\bar{T}_{,im}\right)\Big)\Big)\Big)\Big)\Big),
\end{aligned}
\tag{18}
$$

to the last line

$$
\bar{\Omega}^{-4}\left(2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)\left(2\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{:ij}+\bar{g}^{ei}\bar{R}\bar{T}_{:ei}\right)+\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{R}_{ij}\right)_{,m}
$$

$$
\begin{aligned}
&= \bar{\Omega}^{-4}\Bigg(2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^2 \left(\bar{L}-1\right)\left(2\bar{g}^{ei}\bar{g}^{fj}\bar{R}_{ef}\bar{T}_{:ij} + \bar{g}^{ei}\bar{R}\bar{T}_{:ei}\right)\bar{T}_{,m} \\
&+ 2\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)\Bigg(2\left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj} + \bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{R}_{ef}\bar{T}_{:ij} \\
&+ 2\bar{g}^{ei}\bar{g}^{fj}\left(\bar{R}_{ef,m}\bar{T}_{:ij} + \bar{R}_{ef}\bar{T}_{:ij,m}\right) + \bar{g}^{ei}{}_{,m}\bar{R}\bar{T}_{:ei} + \bar{g}^{ei}\left(\bar{R}_{,m}\bar{T}_{:ei} + \bar{R}\bar{T}_{:ei,m}\right)\Bigg) \\
&+ \left(\bar{g}^{ei}{}_{,m}\bar{g}^{fj} + \bar{g}^{ei}\bar{g}^{fj}{}_{,m}\right)\bar{R}_{ef}\bar{R}_{ij} + \bar{g}^{ei}\bar{g}^{fj}\left(\bar{R}_{ef,m}\bar{R}_{ij} + \bar{R}_{ef}\bar{R}_{ij,m}\right)\Bigg). \qquad (19)
\end{aligned}
$$

4. Asymptotic Monotonicity of K

Thanks to the work in the last section, we are now in a position to determine the monotonic behaviour of K. It is explicit in the below theorem but to be clear - we will be dealing with a regular unphysical metric and hence all unphysical metric components, and their derivatives, will be well behaved at the hypersurface $\bar{T}=0$. Furthermore, this means that theorem 12 does not apply to the AFS and AFEU. It is also important to remember that for a regular unphysical metric (an IPS/IFS or PIU/FIU) $\bar{\Omega}$ is C^3 and as such $\bar{L}_{,m}$ will be well behaved.

Theorem 12 (The K-Prime Theorem) Let $(\mathcal{M},\mathbf{g})$ and $(\bar{\mathcal{M}},\bar{\mathbf{g}})$ be two spacetimes which are related via the conformal structure $\mathbf{g}=\bar{\mathbf{\Omega}}^{\mathbf{2}}(\bar{\mathbf{T}})\bar{\mathbf{g}}$, where $\bar{T}$ is a smooth cosmic time function defined on $(\bar{\mathcal{M}},\bar{\mathbf{g}})$ and $\bar{g}$ is non-degenerate and at least C^2 on an open neighbourhood of $\bar{T}=0$. If

$$\bar{C}^{abcd}\bar{C}_{abcd} \not\equiv 0 \qquad (20)$$

and one of the following conditions are satisfied

i) $\bar{T}\to 0^+$, $\lim\limits_{\bar{T}\to 0^+}\bar{\Omega}=0$ and $\bar{\Omega}$ is positive, C^3 and strictly decreasing on some interval $[-c,0)$,

ii) $\bar{T}\to 0^-$, $\lim\limits_{\bar{T}\to 0^-}\bar{\Omega}=0$ and $\bar{\Omega}$ is positive, C^3 and strictly decreasing on some interval $[-c,0)$,

iii) $\bar{T}\to 0^-$, $\lim\limits_{\bar{T}\to 0^-}\bar{\Omega}=+\infty$ and $\bar{\Omega}$ is positive, C^3 and strictly increasing on some interval $(0,c]$,

then

$$\lim_{\bar{T}\to 0^\pm} K' > 0.$$

If, however

iv) $\bar{T} \to 0^+$, $\lim_{\bar{T}\to 0^+} \bar{\Omega} = +\infty$ and $\bar{\Omega}$ is positive, C^3 and strictly increasing on some interval $(0, c]$

then

$$\lim_{\bar{T}\to 0^-} K' < 0.$$

Proof

For subcases i) and ii) the dominant term in the Ricci scalar's derivative is

$$\left(R^{ef}R_{ef}\right)_{,m} \approx 48\bar{M}^5\bar{\Omega}\left(\bar{L}-1\right)\left(\bar{L}^2-\bar{L}+1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{T}_{,m} \tag{21}$$

because this contains the highest power of $\bar{M} := \frac{\bar{\Omega}'}{\bar{\Omega}^2}$ (which is divergent for these subcases [3]); all other terms are either regular or bounded.

For the subcases iii) and iv) the dominant term of the Ricci scalar's derivative is

$$\left(R^{ef}R_{ef}\right)_{,m} \approx -48\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^5\left(\bar{L}-1\right)\left(\bar{L}^2-\bar{L}+1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{T}_{,m} \tag{22}$$

because it contains the highest power of $\bar{\Omega}'/\bar{\Omega}$‖ and all other terms will be regular or bounded.

While for the Ricci scalar, the dominant term is always going to be

$$R^{ef}R_{ef} \sim 12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right). \tag{23}$$

Initially we consider cases i) and ii). The entropy scalar's derivative, in this case is give by,

$$\begin{aligned} K_{,m} &= \frac{-4\bar{\Omega}^{-5}(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}R^{ef}R_{ef}\bar{T}_{,m} - \bar{\Omega}^{-4}(R^{ef}R_{ef})_{,m}\bar{C}^{abcd}\bar{C}_{abcd}}{\left(R^{ef}R_{ef}\right)^2} \\ &\sim \frac{-4\bar{\Omega}^{-5}(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right))\bar{T}_{,m}}{\left(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)\right)^2} \end{aligned} \tag{24}$$

‖ the behaviour of this function has been well described previously [3]

$$- \frac{\bar{\Omega}^{-4}\left(48\bar{M}^5\bar{\Omega}\left(\bar{L}-1\right)\left(\bar{L}^2-\bar{L}+1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{T}_{,m}\right)\bar{C}^{abcd}\bar{C}_{abcd}}{\left(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)\right)^2} \tag{25}$$

$$= \frac{-(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}\bar{T}_{,m}}{3\bar{\Omega}^5\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)} - \frac{\left(\bar{L}-1\right)\bar{C}^{abcd}\bar{C}_{abcd}\bar{T}_{,m}}{3\bar{\Omega}^7\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^3\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)} \tag{26}$$

$$\sim \frac{\left(1-\bar{L}\right)\bar{C}^{abcd}\bar{C}_{abcd}\bar{T}_{,m}}{3\bar{\Omega}^7\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^3\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)} \tag{27}$$

$$= \frac{1}{\bar{\Omega}}\frac{1}{\frac{\bar{\Omega}'}{\bar{\Omega}}}\frac{\bar{C}^{abcd}\bar{C}_{abcd}\left(1-\bar{L}\right)\bar{T}_{,m}}{3\bar{\Omega}^6\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^2\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)}. \tag{28}$$

The reason for writing it in this form becomes clear when we note that $\lim_{\bar{T}\to 0^\pm}\bar{\Omega}(\bar{T})\to 0$, $\bar{\lambda}<1$. Therefore the sign of $K_{,m}$ solely depends on the signs of $\bar{\Omega}$ and $\bar{\Omega}'/\bar{\Omega}$ because all other terms are positive and nonzero.

The reader will recall that for case i), $\lim_{\bar{T}\to 0^+} 1/\bar{\Omega}\to\infty^+$ and $\lim_{\bar{T}\to 0^+}\bar{\Omega}'/\bar{\Omega}\to+\infty$. Hence the above is always positive and so is the entropy scalar's derivative.

For case ii), $\lim_{\bar{T}\to 0^-} 1/\bar{\Omega}\to\infty^-$ and $\lim_{\bar{T}\to 0^-}\bar{\Omega}'/\bar{\Omega}\to-\infty$ and hence the above is positive and so is $K_{,m}$.

We turn to cases iii) and iv) now. The entropy scalar's derivative is

$$K_{,m} = \frac{-4\bar{\Omega}^{-5}(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}R^{ef}R_{ef}\bar{T}_{,m}-\bar{\Omega}^{-4}(R^{ef}R_{ef})_{,m}\bar{C}^{abcd}\bar{C}_{abcd}}{\left(R^{ef}R_{ef}\right)^2} \tag{29}$$

$$\sim \frac{-4\bar{\Omega}^{-5}(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right))\bar{T}_{,m}}{\left(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)\right)^2} + \frac{\bar{\Omega}^{-4}\left(48\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^5\left(\bar{L}-1\right)\left(\bar{L}^2-\bar{L}+1\right)\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\bar{T}_{,m}\right)\bar{C}^{abcd}\bar{C}_{abcd}}{\left(12\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^4\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^2\left(\bar{L}^2-\bar{L}+1\right)\right)^2} \tag{30}$$

$$
\begin{aligned}
&= \frac{-\bar{\Omega}^{-5}(\bar{C}^{abcd}\bar{C}_{abcd})_{,m}\bar{T}_{,m}}{3\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{4}\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^{2}\left(\bar{L}^{2}-\bar{L}+1\right)} \\
&+ \frac{\bar{\Omega}^{-4}\frac{\bar{\Omega}'}{\bar{\Omega}}\bar{C}^{abcd}\bar{C}_{abcd}\bar{T}_{,m}}{3\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{4}\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^{2}\left(\bar{L}^{2}-\bar{L}+1\right)} \qquad (31) \\
&\sim \frac{\bar{C}^{abcd}\bar{C}_{abcd}\bar{T}_{,m}}{3\bar{\Omega}^{4}\left(\frac{\bar{\Omega}'}{\bar{\Omega}}\right)^{3}\left(\bar{g}^{ei}\bar{T}_{,e}\bar{T}_{,i}\right)^{2}\left(\bar{L}^{2}-\bar{L}+1\right)} \qquad (32)
\end{aligned}
$$

As we saw before, this mathematical form is helpful because $\lim_{\bar{T}\to 0^{\pm}} \bar{\Omega}^{4}(\bar{T}) \to +\infty$, $\bar{\lambda} > 1$. So the sign of $K_{,m}$ solely depends on the sign of $\bar{\Omega}'/\bar{\Omega}$ because all other terms are positive and nonzero.

For case iii) $\lim_{\bar{T}\to 0^{+}} \bar{\Omega}'/\bar{\Omega} \to -\infty$ and hence the above is negative.

For case iv) $\lim_{\bar{T}\to 0^{-}} \bar{\Omega}'/\bar{\Omega} \to +\infty$ and hence the above is positive. $\square$

This now completes the proof. In order to guide the reader in visualising this behaviour, we present three representations of the monotonicity of K. The first represents cases i) and iii), the second is case ii) and the last is case iv).

Figure 5. A representation of how K would behave if it were monotonically increasing away from $\bar{T} = 0$

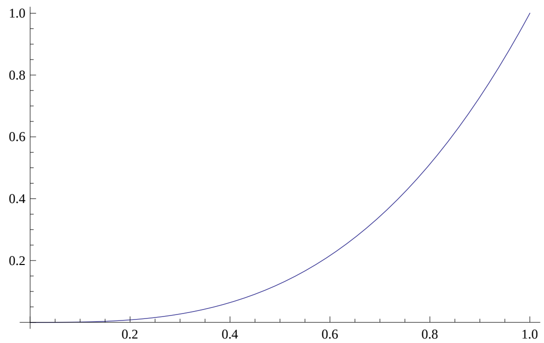


Figure 6. A representation of how K would behave if it were monotonically decreasing toward $\bar{T} = 0$

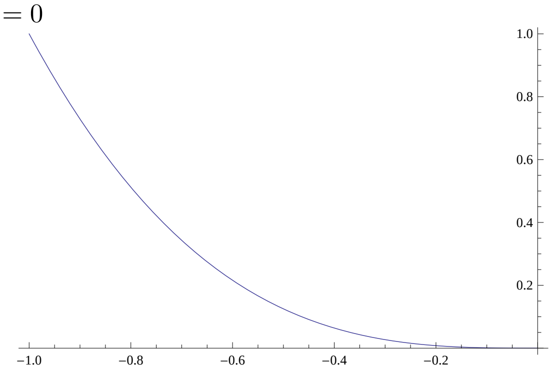
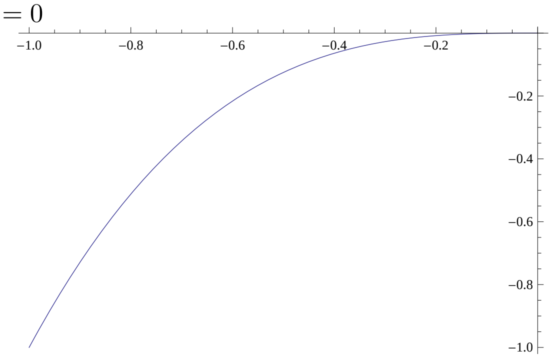


Figure 7. A representation of how K would behave if it were monotonically increasing toward $\bar{T} = 0$



This is fundamentally important because it means for initial isotropic structures, their measure of gravitational entropy will increase away from zero and for final isotropic states, their gravitational entropy will decrease towards zero. What we want to ascertain now is how K behaves for anisotropic future states. At this stage, all example cosmologies that admit an AFEU or AFS have $K > 0$ but we have not been able to prove this in all generality. As the direction of this study will be somewhat different

to this paper, we defer this discussion to an upcoming paper apart from the following remarks.

All observational evidence indicates that, at least locally, entropy is ever increasing and if Quiescent Cosmology is to be consistent with observational evidence (as well as Penrose's WCH) then the AFEU and AFS should have a measure of gravitational entropy that is nonzero. This will serve to demonstrate that a universe that begins with an isotropic structure and ends in an anisotropic state will have a net increase of gravitational entropy. This will not serve to demonstrate monotonicity in the intermediate region as gravitational entropy may be oscillatory in nature during this region but it will show a net increase.

5. Conclusions and Further Outlook

The work in this paper is pivotal to prove not only the physical plausibility of an IPS but also to demonstrate that the IPS is truly compatible with the WCH. We have been able to show that the gravitational entropy scalar will, in a local neighbourhood of the IPS at $\bar{T} = 0$, monotonically increase for non conformally flat spacetimes. This is in direct agreement with Penrose's conjecture regarding the dominance of the Weyl scalar.

Furthermore, we have also been able to prove that, if the Universe did not start with a Big Bang but rather was a uniform distribution of matter, corresponding to a PIU then this too has zero gravitational entropy that monotonically increases as cosmic time increases. Although classical General Relativity seems to predict that the Universe started with a Big Bang, it is reassuring nevertheless, that Quiescent Cosmology and the WCH is compatible with this structure.

If the Universe were to end in an isotropic singularity then the gravitational entropy will be locally monotonically increasing towards zero. This seems to indicate that K would obtain a maximum (negative) value before increasing to zero. This is somewhat similar to the case when the Universe ends as an FIU because in this case the entropy

scalar decreases monotonically as the FIU is approached. Both of these scenarios indicate that at some stage prior to the isotropic end, the Universe had a maximum, nonzero value of gravitational entropy and that it will tend to zero in the future. This is not compatible with the second law of thermodynamics but it means that if the end of the Universe is going to be isotropic then it means gravitational entropy will have to decrease from some finite maximal value.

As mentioned at the end of our main results section, the obvious extension for this type of work is to consider our anisotropic futures and see if their gravitational entropy scalar's are monotonically increasing as they are approached. As will be seen in future papers, the problems caused by the degenerate nature of the AFEU and AFS will force us to address the question of gravitational entropy in a different manner.

References

[1] J.D. Barrow. Quiescent Cosmology. *Nature*, 272:211–215, 1978.

[2] S. W. Goode and J. Wainwright. Isotropic Singularities in Cosmological Models. *Classical and Quantum Gravity*, 2:99–115, 1985.

[3] P. A. Höhn and S. M. Scott. Encoding Cosmological Futures with Conformal Structures. *Classical and Quantum Gravity*, 26:035019, 2009.

[4] R. Penrose. *Singularities and Time-Asymmetry*, chapter 12, pages 581–638. Cambridge University Press, 1979.

[5] C. W. Misner. Transport Processes in the Primordial Fireball. *Nature*, 214:40–41, 1967.

[6] S.M. Scott. The General Vorticity Result. (In preparation).

[7] P. Threlfall. *Expanding the Quiescent Cosmology Framework*. PhD thesis, The Australian National University, 2013.

[8] S. W. Goode, A. A. Coley, and J. Wainwright. The Isotropic Singularity in Cosmology. *Classical and Quantum Gravity*, 9:445–455, 1992.

[9] J. D. Barrow and S. Hervik. The Weyl Tensor in Spatially Homogeneous Cosmological Models. *Classical and Quantum Gravity*, 19:5173–5193, 2002.

[10] W. C. Lim, C. Uggla, and J. Wainright. Asymptotic Silence-Breaking Singularities. *General Relativity and Gravitation*, 23:2607–2630, 2006.

[11] K. P. Tod. Isotropic Singularities and the $\gamma = 2$ Equation of State. *Classical and Quantum Gravity*, 7:L13–L16, 1990.